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ECCENTRIC GRAPHS

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For any graph G we define the eccentric graph G_e on the same set of vertices, by joining two vertices in G_e if and only if one of the vertices has maximum possible distance from the other. The following results are given in this paper:

(1) A few general properties of eccentric graphs.
(2) A characterization of graphs G with $G_e = K_p$ and with $G_e = pK_2$.
(3) A solution of the equation $G_e = \bar{G}$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph. The *eccentricity* $e(v)$ of a vertex in $V(G)$ is defined by

$$e(v) = \max_{u \in V(G)} d(u, v),$$

where $d(u, v)$ stands for the length of the shortest path in G between u and v. In case G is disconnected and u and v belong to different components, we set $d(u, v) = +\infty$. We denote by $G_e = (V(G_e), E(G_e))$ the *eccentric graph* based on G. The vertex set $V(G_e)$ is identical to $V(G)$ and

$$uv \in E(G_e) \Leftrightarrow d(u, v) = \min(e(u), e(v)).$$

A similar graph, called the "furthest neighbour graph", was introduced by Shamos [7]. Its vertex set is a set of points in the plane, and the distance between two vertices is their Euclidean distance. Two vertices are joined if either one is the "furthest neighbour" of the other. Extremal properties of this graph are studied in [2].

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A *central vertex* of a graph G is a vertex v with the property that the maxin distance between v and any other vertex is as small as possible, this distance be called the *radius*, denoted by $r(G)$. That is $r(G) = \min_v \max_w d(v, w)$. The *dia*n*ter* of G denoted $\text{diam}(G)$ is defined by $\text{diam}(G) = \max_v \max_w d(v, w)$.

A graph is a *self-center* [4] or *r-equi-eccentric* (or briefly *r-equi*) [1] $e(v) = r(G) = \text{diam}(G)$ for all vertices $v \in V(G)$. If $S \subset V(G)$, then we say th $e(S) = i$ if $e(v) = i$ for all $v \in S$. We denote by $N(v)$ the *neighbourhood* of a vert v of G consisting of the vertices in G adjacent to v. The *closed* neighbourho $N[v]$ of v is defined by $N[v] = N(v) \cup \{v\}$.

All other definitions and notation used in this paper may be found in [3] or [5

We first present a few general results on eccentric graphs. In the next sectio we examine some special eccentric graphs: K_p and pK_2. The third section devoted to examining the equations $G_e = G$ and $G_e = \bar{G}$.

Proposition 1.1. *If $r(G) > 2$, then $(G + K_n)_e = \bar{G} + K_n$, $n \geq 1$.*

Proof. Consider any vertex $u \in K_n$. Since u is adjacent to all vertices in $G + K_n$, will be adjacent to all vertices in $(G + K_n)_e$. Consider any vertex $u \in G$. Sinc $r(G) > 2$, there exists some vertex $v \in G$ such that u is not adjacent to v. Thus has eccentricity 2 in $G + K_n$ and u and v are adjacent in $(G + K_n)_e$ to precisel those vertices which it is non-adjacent to in G. The proposition follows. □

Corollary 1.1. *Let G be any graph of order p. Then there exists an eccentric grap H of order $p + 2$ such that G is an induced subgraph of H.*

Proof. Set $H = (G + K_1) + K_1$ and apply Proposition 1.1. □

Note that the above corollary suggests that it is impossible to characteriz furthest neighbour graphs in terms of forbidden subgraphs.

2. Some special eccentric graphs

In this section we examine and characterize those graphs which have eccentric graphs isomorphic to either K_p or pK_2.

Theorem 2.1. *For any connected graph G of order p, $G_e = K_p$ if and only if for al $v \in V(G)$ either $e(v) = 1$ or $e(N(v)) = 1$.*

Proof. Suppose $G_e = K_p$ and the condition is false. Because G is connected there exist vertices v, w of G such that $e(v) > 1$, $e(w) > 1$ and $w \in N(v)$. But this implies that v is not a furthest vertex of w and w is not a furthest vertex of v. Thus edge vw is not contained in G_e which is a contradiction. On the other hand, suppose the condition is satisfied and let v be any vertex of G. Then if $e(v)=1$, all other vertices are furthest vertices of v, hence all edges from vertex v are present in G_e. If $e(v)>1$, let w be any vertex in $N(v)$. The condition states that $e(w)=1$, so v is a furthest vertex of w and edge vw is contained in G_e. If $w \notin N(v)$, then $d(v, w)=2$, since the condition implies that $r(G)=1$. In this case w is a furthest vertex of v so edge vw is present in G_e. Therefore $G_e=K_p$. □

A graph G is called *radius-critical* if $r(G-v)=r(G)-1$ for all $v \in V(G)$. We now show that radius critical graphs are precisely those graphs whose eccentric graph is a perfect matching. We begin with a lemma.

Lemma 2.1. *If $G_e=\bigcup_{i=1}^n K_{p_i}$, where $\sum_{i=1}^n p_i=p$, $n \geqslant 2$, then G is equi-eccentric.*

Proof. Since $i \geqslant 2$, G must be connected and $r(G)>1$. Set $k=\operatorname{diam}(G)$ and let x and y be any pair of vertices in G such that $d(x, y)=k$. Let z be any vertex in $N(x)$. Since xz is not an edge of G_e, neither is yz. It follows that $d(y, z) \leqslant k-1$. The triangle inequality then implies that $d(y, z)=k-1$. The furthest vertex of z must thus be at distance k from z, hence $e(z)=k$. Thus $e(N(x))=k$. A similar argument shows that $e(N(y))=k$. We can repeat a similar argument for vertices in $N(x) \cup N(y)$. Since G is connected this shows that $e(V(G))=k$, so G is k-equi. □

Theorem 2.2. *If G is a graph of order $2p$ then $G_e=pK_2$ if and only if G is radius critical.*

Proof. The theorem is trivial for $p=1$. So we may assume that $p \geqslant 2$ and hence that G is connected. Set $k=\operatorname{diam}(G)$. First suppose that $G_e=pK_2$. We may apply the lemma with $p_i=2$ for all i, to see that G is k-equi. Let u be any vertex and let v be the unique vertex adjacent to u in G_e. Let d' denote distance in $G-u$. Then $d(v, x)=d'(v, x) \leqslant k-1$ for all $x \in v(G-u)$, and $r(G-u) \leqslant k-1$. The eccentricities of the other vertices in $G-u$ cannot decrease, thus $r(G-u)=k-1$.

On the other hand suppose that G is radius critical. Set $r=r(G)$. Let u be any vertex in V and let v be a vertex in the center of $G-u$. Then letting d' denote distances in $G-u$, we have (i) $d(v, x) \leqslant d'(v, x) \leqslant r-1$ for all $x \in V(G)$, $x \neq u, v$, (ii) $d(u, v)=r$.

Thus u is the unique furthest vertex of v in G. We will show that v is also the unique farthest vertex of u. Let w be any other vertex such that $d(w, u)=r$. In view of (ii) the shortest path from w to u does not include vertex v. But then the distance from w to u in $G-v$ is also r. This contradicts the fact that G is radius critical. Thus uv is an isolated edge in G_e and hence $G_e=pK_2$. □

We remark that graphs satisfying the condition of Theorem 2.2 are referred to as *unique eccentric point* (u.e.p.) graphs in [6]. Further properties of these graphs are presented in this reference. In particular, u.e.p. graphs that are self-centr are characterized.

3. The equations $G_e = \bar{G}$ and $G_e = G$

In this section we completely characterize all graphs whose complements, are isomorphic to their eccentric graph. We also give partial results for the characterization of graphs isomorphic to their eccentric graphs.

Before stating the main result, we need the following notation. Let $S_i = \{v \in V(G) \mid e(v) = i\}$, $i = 1, 2, \ldots$. Then we can completely characterize those graphs whose eccentric graph is isomorphic to its complement.

Theorem 3.1. *$G_e = \bar{G}$ if and only if $S_i = \emptyset$, $i = 1, 4, 5, 6, \ldots$ and no two vertices in S_3 have a common nieghbour.*

Proof. We first show the necessity of the conditions. Suppose $S_1 \neq \emptyset$. Then $\bar{G}$ has an isolated vertex and so $G_e \neq \bar{G}$. We may thus assume that $r(G) > 1$. In this case it is an easy observation that $G_e \supseteq \bar{G}$. This implies that if G_e is isomorphic to $\bar{G}$ it must be isomorphic under the identity mapping. Hence if vertices u and v are adjacent in $\bar{G}$ but not in G_e, then $\bar{G}$ is not isomorphic to G_e. Suppose next that $S_i \neq \emptyset$ for some i greater than or equal to 4. Then there exists a shortest path $u_0 u_1 u_2 \ldots u_i$ in G. Now $e(u_1) \geq i-1$ and $e(u_{i-1}) \geq i-1$. However $d(u_1, u_{i-1}) = i-2$ and so $u_1 u_{i-1}$ is an edge in $\bar{G}$ but not in G_e. Finally suppose $u, v \in S_3$ and $d(u, v) = 2$. It follows immediately that u and v are not adjacent in G_e but are adjacent in G.

We now turn to the sufficiency of the conditions. Suppose firstly that $S_3 = \emptyset$. Then G is 2-equi and hence each vertex in G_e is adjacent to all vertices at distance 2, so $G_e = \bar{G}$. Suppose $S_3 \neq \emptyset$. Consider any vertex $u \in S_3$. Let T_i, $i = 1, 2, 3$ be the sets of vertices at distance i from u. By the conditions $T_2 \cap S_3 = \emptyset$. Therefore $e(T_2) = 2$. Thus u will be connected to all vertices in $T_2 \cup T_3$ in G_e. These are precisely the neighbours of u in $\bar{G}$. Consider any vertex v in S_2. Again, v will be adjacent in G_e to all vertices at distance 2 in G, which are precisely its neighbours in $\bar{G}$. Therefore $G_e = \bar{G}$ and the theorem is proved. $\square$

As the next result shows, the conditions of the theorem are not overly restrictive. In fact, 'almost all' graphs satisfy $G_e = \bar{G}$. A random labelled graph of order p is a graph on p vertices whose edges are chosen independently with probability 1/2.

Theorem 3.2. *If G is a randomly selected labelled graph of order p, then*

$$\mathrm{Prob}(G_e = \bar{G}) \geq 1 - \frac{p^2}{2}\left(\frac{3}{4}\right)^{p-2}.$$

Proof. We will show that 'almost all' graphs are 2-equi. We first observe that $d(u, v) \geqslant 3$ if and only if u, v are non-adjacent and have no common neighbour. Thus

$$\mathrm{Prob}(d(u, v) \geqslant 3) = \tfrac{1}{2}(\tfrac{3}{4})^{p-2},$$

and

$$\mathrm{Prob}(\exists u, v \text{ such that } d(u, v) \geqslant 3) \leqslant \sum_{u, v} \mathrm{Prob}(d(u, v) \geqslant 3) = \frac{1}{2} \cdot \binom{p}{2} \left(\frac{3}{4}\right)^{p-2}.$$

Similarly,

$$\mathrm{Prob}(\exists v \text{ such that } e(v) = 1) \leqslant \sum_{v} \mathrm{Prob}(e(v) = 1) = p(\tfrac{1}{2})^{p-1}.$$

It is easily verified that

$$\frac{1}{2} \cdot \binom{p}{2} \left(\frac{3}{4}\right)^{p-2} + p\left(\frac{1}{2}\right)^{p-1} \leqslant \frac{p^2}{2} \left(\frac{3}{4}\right)^{p-2}$$

and hence

$$\mathrm{Prob}(G_e = \bar{G}) \geqslant P(G \text{ is 2-equi}) \geqslant 1 - \frac{p^2}{2} \left(\frac{3}{4}\right)^{p-2}. \quad \square$$

We turn now to the equation $G_e = G$. This appears to be more difficult to solve and we present only partial results. Recall that S_i denotes the set of vertices in G of eccentricity i.

Theorem 3.3. *If $r(G) = 1$, then $G_e = G$ if and only if $\langle V - S_1 \rangle_G$ is self-complementary.*

Proof. Suppose $v \in S_1$. Then v has eccentricity 1 in G_e. Thus if $v, w \in V - S_1$ they are adjacent in G if and only if they are non-adjacent in G_e. Therefore any isomorphism from G_e to G must map vertices in $V - S_1$ to vertices in $V - S_1$, hence this induced subgraph of G must be self-complementary. $\square$

We remark that $G_e = \bar{G}$ is possible only under the identity isomorphism, but this is not necessarily the case for $G_e = G$ (see Fig. 1). When $r(G) > 1$, we present only the following necessary condition.

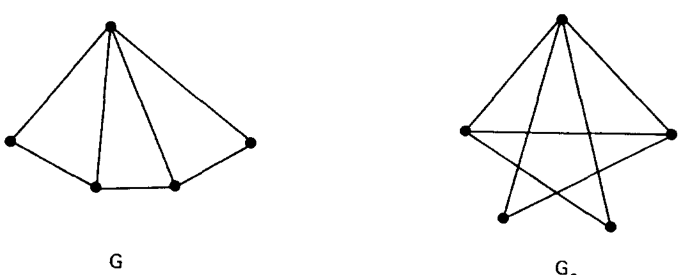

Fig. 1. $G = G_e$.

Theorem 3.4. *If $r(G)>1$ and $G=G_e$, then $d_i+d_{p-i+1}\leq p-1$, where $d_1\leq d_2\leq \cdots \leq d_p$ is the degree sequence of G.*

Proof. As previously remarked, $r(G)>1$ implies that $G_e\subseteq \bar{G}$. Therefore the degree sequence of G_e is majorized by the degree sequence of $\bar{G}: p-1-d_p, p-1-d_{p-1},\ldots,p-1-d_1$. Since G and G_e have the same degree sequence, the theorem follows. □

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